

# Interior Regularity for a generalized Abreu Equation <sup>☆</sup>

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## Abstract

We study a generalized Abreu Equation in  $n$ -dimensional polytopes and derive interior estimates of solutions under the assumption of the uniform  $K$ -stability.

*Keywords:* Interior estimates, generalized Abreu Equation.

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## 1. Introduction

The existence of extremal and constant scalar curvature is a central problem in Kähler geometry. In a series of papers [10], [11], [12], and [13], Donaldson studied this problem on toric manifolds and proved the existence of metrics of constant scalar curvatures on toric surfaces under an appropriate stability condition. Later on in [6] and [7], Chen, Li and Sheng proved the existence of metrics of prescribed scalar curvatures on toric surfaces under the uniform stability condition.

It is important to generalize the results of Chen, Li and Sheng to more general Kähler manifold. This is one of a sequence of papers, aiming at generalizing the results of Chen, Li and Sheng to homogeneous toric bundles.

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The primary goal of this paper is to study the following nonlinear fourth-order partial differential equation for an  $n$ -dimensional convex function  $u$

$$\frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j} = -A. \quad (1.1)$$

Here,  $\mathbb{D} > 0$  and  $A$  are two given smooth functions on  $\bar{\Delta}$  and  $(u^{ij})$  is the inverse of the Hessian matrix  $(u_{ij})$ . The equation (1.1) was introduced by Donaldson [14] in the study of the scalar curvature of toric fibration, see also [23] and [21]. In [20] the authors also derived this PDE in the study of the scalar curvature of homogeneous toric bundles. We call (1.1) a generalized Abreu Equation. The main result is the following interior estimate

**Theorem 1.1.** *Let  $\Delta$  be a bounded open polytope in  $\mathbb{R}^n$  and  $\mathbb{D} > 0$ ,  $A$  be two smooth functions on  $\bar{\Delta}$ . Suppose  $(\Delta, \mathbb{D}, A)$  is uniformly  $K$ -stable and  $u$  is a solution in  $\mathbf{S}_{p_o}$  of the equation (1.1). Then, for any  $\Omega \subset\subset \Delta$ , any nonnegative integer  $k$  and any constant  $\alpha \in (0, 1)$ ,*

$$\|u\|_{C^{k+3,\alpha}(\Omega)} \leq C,$$

where  $C$  is a positive constant depending only on  $n, k, \alpha, \Omega, \mathbb{D}, \|A\|_{C^k(\bar{\Delta})}$  and  $\lambda$  in the uniform  $K$ -stability.

A equivalent statement of Theorem 1.1 is the following

**Theorem 1.2.** *Suppose that  $(\Delta, \mathbb{D}, A)$  is uniformly  $K$ -stable and that  $\{A^{(k)}\}$  is a sequence of smooth functions in  $\bar{\Delta}$  such that  $A^{(k)}$  converges to  $A$  smoothly in  $\bar{\Delta}$ . Assume  $u^{(k)} \in \mathbf{S}_{p_o}$  is a sequence of solutions of the generalized Abreu Equation*

$$\sum_{i,j} \frac{\partial^2 (\mathbb{D} u^{(k)ij})}{\partial \xi_i \partial \xi_j} = -A^{(k)} \mathbb{D} \quad \text{in } \Delta. \quad (1.2)$$

*Then there is a subsequence, still denoted by  $u^{(k)}$ , such that  $u^{(k)}$  converges smoothly, in any compact set  $\Omega \subset \Delta$ , to some smooth and strictly convex function  $u$  in  $\Delta$ .*

The main ideal of the proof is following:

Note that, as Donaldson pointed out that, the uniform stability of  $(\Delta, \mathbb{D}, A)$  implies that there is a subsequence, still denoted by  $u^{(k)}$ , locally uniformly

converging to  $u$  in  $\Delta$ . The key point is to prove that  $u$  is smooth and strictly convex. We consider the Legendre transform  $f^{(k)}$  of  $u^{(k)}$ . Then  $f^{(k)}$  satisfy the PDE

$$-\sum_{i,j} f^{ij} \frac{\partial^2(\log \mathbb{F})}{\partial x_i \partial x_j} - \sum_{i,j} f^{ij} \frac{\partial(\log \mathbb{F})}{\partial x_i} \frac{\partial(\log \mathbb{D})}{\partial x_j} = A. \quad (1.3)$$

In Section 3, we derive an uniform lower bound and an uniform upper bound of the determinants of the Hessian of  $f^{(k)}$ . We can not directly apply the Caffarelli and Gutiérrez theory to the PDE (1.3). We prove a convergence theorem for this PDE in Section 4. Then Theorem 1.2 follows.

## 2. Uniform stability

Let  $\Delta$  be a Delzant polytope in  $\mathbb{R}^n$ ,  $c_k$  be a constant and  $h_k$  be an affine linear function in  $\mathbb{R}^n$ ,  $k = 1, \dots, d$ . Suppose that  $\Delta$  is defined by linear inequalities  $h_k(\xi) - c_k > 0$ , for  $k = 1, \dots, d$ , where each  $h_k(\xi) - c_k = 0$  defines a facet of  $\Delta$ . Write  $\delta_k(\xi) = h_k(\xi) - c_k$  and set

$$v(\xi) = \sum_k \delta_k(\xi) \log \delta_k(\xi). \quad (2.1)$$

This function was first introduced by Guillemin [16]. It defines a Kähler metric on the toric variety defined by  $\Delta$ . We introduce several classes of functions. Set

$$\begin{aligned} \mathcal{C} &= \{u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and smooth on } \Delta\}, \\ \mathbf{S} &= \{u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and } u - v \text{ is smooth on } \bar{\Delta}\}, \end{aligned}$$

where  $v$  is given in (2.1). For a fixed point  $p_o \in \Delta$ , we consider

$$\begin{aligned} \mathcal{C}_{p_o} &= \{u \in \mathcal{C} : u \geq u(p_o) = 0\}, \\ \mathbf{S}_{p_o} &= \{u \in \mathbf{S} : u \geq u(p_o) = 0\}. \end{aligned}$$

We say functions in  $\mathcal{C}_{p_o}$  and  $\mathbf{S}_{p_o}$  are *normalized* at  $p_o$ . Let

$$\begin{aligned} \mathcal{C}_* &= \{u | \text{there exist a constant } C > 0 \text{ and a sequence of } \{u^{(k)}\} \text{ in } \mathcal{C}_{p_o} \\ &\quad \text{such that } \int_{\partial \Delta} u^{(k)} \mathbb{D} d\sigma < C \text{ and } u^{(k)} \text{ locally uniformly converges to} \\ &\quad u \text{ in } \Delta\}. \end{aligned}$$

For any  $u \in \mathcal{C}_*$ , define  $u$  on boundary as

$$u(q) = \lim_{\Delta \ni \xi \rightarrow q} u, \quad q \in \partial\Delta.$$

Let  $P > 0$  be a constant, we define

$$\mathcal{C}_*^P = \{u \in \mathcal{C}_* \mid \int_{\partial\Delta} u \mathbb{D} d\sigma \leq P\}.$$

Following [21] we consider the functional

$$\mathcal{F}_A(u) = - \int_{\Delta} \log \det(u_{ij}) \mathbb{D} d\mu + \mathcal{L}_A(u), \quad (2.2)$$

where

$$\mathcal{L}_A(u) = \int_{\partial\Delta} u \mathbb{D} d\sigma - \int_{\Delta} A u \mathbb{D} d\mu. \quad (2.3)$$

$\mathcal{F}_A$  is called the Mabuchi functional and  $\mathcal{L}_A$  is closely related to the Futaki invariants. The Euler-Lagrangian equation for  $\mathcal{F}_A$  is (1.1). It is known that, if  $u \in \mathbf{S}$  satisfies the equation (1.1), then  $u$  is an absolute minimizer for  $\mathcal{F}_A$  on  $\mathbf{S}$ .

**Definition 2.1.** Let  $\mathbb{D} > 0$  and  $A$  be two smooth functions on  $\bar{\Delta}$ . Then,  $(\Delta, \mathbb{D}, A)$  is called *uniformly K-stable* if the functional  $\mathcal{L}_A$  vanishes on affine-linear functions and there exists a constant  $\lambda > 0$  such that, for any  $u \in \mathcal{C}_{p_o}$ ,

$$\mathcal{L}_A(u) \geq \lambda \int_{\partial\Delta} u \mathbb{D} d\sigma. \quad (2.4)$$

We also say that  $\Delta$  is  $(\mathbb{D}, A, \lambda)$ -stable.

**Remark 2.2.** The conditions in Definition 2.1 are exactly the contents of Condition 1 [21]. Following Donaldson we call it the *uniform K-stability*.

Using the same method in [9] we immediately get

**Theorem 2.3.** *If the equation (1.1) has a solution in  $\mathbf{S}$ , then  $(\Delta, \mathbb{D}, A)$  is uniform K-stable.*

Namely, the uniform K-stability is a necessary condition for existing a solution of (1.1) in  $\mathbf{S}$ . We pose the

**Question 2.4.** Let  $\Delta \subset \mathbb{R}^n$  be a Delzant polytope,  $\mathbb{D} > 0$  and  $A$  be two smooth functions on  $\bar{\Delta}$ . Does the uniform K-stability of  $(\Delta, \mathbb{D}, A)$  imply that the equation (1.1) has a solution in  $\mathbf{S}$ ?

Assume that  $v \in \mathbf{S}_{p_o}$  is the solution of the equation (1.1), and  $u$  is a convex function. For any segment  $I \subset\subset \Delta$ ,  $u$  defines a convex function  $w := u|_I$  on  $I$ . It defines a Monge-Ampere measure on  $I$ , we denote this by  $N$ . The key point of the proof in [9] is the following lemma.

**Lemma 2.5.** *Let  $u \in \mathcal{C}_*^P$  and  $u^{(k)} \in \mathcal{C}$  locally uniformly converges to  $u$ . If  $N(I) = m > 0$ , then*

$$\mathcal{L}_A(u^{(k)}) > \tau m$$

*for some positive constant  $\tau$  independent of  $k$ .*

In our present case this lemma still holds due to  $C^{-1} \leq \mathbb{D} \leq C$  for some constant  $C > 0$ . For reader's convenience we give the proofs here.

**Proof of Lemma 2.5.** Let  $p$  be the midpoint of  $I$ . We choose coordinate system  $\{0, \xi\}$  such that  $p$  is the origin,  $I$  is on the  $\xi_1$  axis and  $I = (-a, a)$ . Set  $I_\epsilon = [-a + \epsilon, a - \epsilon]$ . By choosing  $\epsilon$  small we can assume that

$$N(I_\epsilon) \geq \frac{3m}{4}. \quad (2.5)$$

Suppose that there is a Euclidean ball  $B := B_{\epsilon_o}(0)$  in  $\xi_1 = 0$  plane such that  $I \times B \subset\subset \Delta$ . Suppose that  $u$  is a limit of a sequence  $u^{(k)} \in \mathcal{C}$ . Then  $u^{(k)}$  converges to  $u$  uniformly on  $I \times B$ . We have

$$\mathcal{L}_A(u^{(k)}) = \int_{\Delta} v^{ij} u_{ij}^{(k)} \mathbb{D} d\mu. \quad (2.6)$$

Consider the functions

$$w_\xi^{(k)}(\xi_1) = u^{(k)}(\xi_1, \xi), \quad \xi_1 \in I, \xi \in B.$$

We denote by  $N_\xi^{(k)}$  the Monge-Ampere measure on  $I$  induced by  $w_\xi^{(k)}$ . We claim that there exists a small  $B$  and large  $K$  such that for any  $\xi \in B$ ,  $k > K$

$$N_\xi^{(k)}(I) \geq m/2. \quad (2.7)$$

In fact, if not, then there exists a subsequence of  $k$ , still denote by  $k$ , and a sequence of  $\xi_k \in B$  with  $\xi_k \rightarrow 0$  such that  $N_{\xi_k}^{(k)}(I) < m/2$ . However, by the weakly convergence of Monge-Ampere measure, we have

$$N(I_\epsilon) \leq \lim_{k \rightarrow \infty} N_{\xi_k}^{(k)}(I) \leq m/2,$$

this contradicts (2.5).

On the other hand, the eigenvalues of  $v^{ij}$  are bounded below in  $I \times B$ , let  $\delta$  be the lower bound. Then

$$\begin{aligned} \mathcal{L}_A(u^{(k)}) &\geq \int_{I \times B} v^{ij} u_{ij}^{(k)} \mathbb{D} d\mu \geq \frac{\delta}{C} \int_{I \times B} \text{Trace}(u_{ij}^{(k)}) d\mu \\ &\geq \frac{\delta}{C} \int_{I \times B} u_{11}^{(k)} d\mu = \frac{\delta}{C} \int_B N_\xi^{(k)}(I) d\xi \geq \frac{m\delta}{2C} \text{Vol}(B). \end{aligned}$$

This completes the proof of Lemma 2.5.

Then by the same method as in [9] we can prove Theorem 2.3.

### 3. Estimates of the Determinant

Set

$$\mathbb{F} := \frac{\mathbb{D}}{\det(u_{ij})}, \quad U^{ij} = \det(u_{kl}) u^{ij}. \quad (3.1)$$

Since  $\sum_i U_i^{ij} = 0$ , the generalized Abreu Equation (1.1) can be written in terms of  $(\xi, u)$  as

$$-\sum_{i,j} U^{ij} \frac{\partial^2 \mathbb{F}}{\partial \xi_i \partial \xi_j} = A \mathbb{D}. \quad (3.2)$$

Through the normal map  $\nabla u$  we can view  $\mathbb{D}$  as function in  $x$ . In terms of  $(x, f)$  the PDE (3.11) can be written as

$$-\sum_{i,j} f^{ij} \frac{\partial^2 (\log \mathbb{F})}{\partial x_i \partial x_j} - \sum_{i,j} f^{ij} \frac{\partial (\log \mathbb{F})}{\partial x_i} \frac{\partial (\log \mathbb{D})}{\partial x_j} = A. \quad (3.3)$$

#### 3.1. The lower bound of the determinant

The following Lemma is proved in [21] for toricfibration. It can be extend directly to the generalized Abreu Equation (1.1).

**Lemma 3.1.** *Let  $\Delta$  be a bounded open polytope in  $\mathbb{R}^n$  and  $\mathbb{D} > 0$ ,  $A$  be two smooth functions on  $\bar{\Delta}$ . Let  $u \in \mathcal{C}$  be a strictly convex function satisfying the generalized Abreu Equation (1.1). Suppose that  $\mathbb{F} = 0$  on  $\partial\Delta$ . Then*

$$\det(u_{ij}) \geq C_1(\sup_{\Delta} A)^{-n}$$

*everywhere in  $\Delta$ , where  $C_1$  is a constant depending on  $n, \mathbb{D}$  and  $\Delta$ .*

In the following we derive a more stronger estimate than Lemma 3.1, which will be used in our next papers. First we prove a preliminary lemma.

**Lemma 3.2.** *Let  $\Delta$  be a bounded open polytope. Suppose that  $\mathbb{F} = 0$  on  $\partial\Delta$ . Let  $E$  be an edge of  $\Delta$ . Suppose that  $E$  is given by  $\xi_1 = 0$ . Set*

$$v(\alpha, \beta, C) = -\xi_1^\alpha (C - \xi_1)^\beta \left( C - \sum_{j=2}^n \xi_j^2 \right)^\beta,$$

*where  $\alpha, \beta, C$  are constants. Then for any  $\frac{1}{2n} \leq \alpha, \beta \leq 1 - \frac{1}{2n}$ , there exists constants  $C, C_1 > 0$  depending only on  $n$  and  $\text{diam}(\Delta)$  such that  $v$  is strictly convex and*

$$\det(v_{ij}) > C_1(\epsilon_0)\xi_1^{n\alpha-2}. \quad (3.4)$$

*Proof.* Choose  $C > 0$  large such that

$$\Delta \subset \left\{ \xi \mid \xi_1 \leq \frac{C}{m} \right\} \cap \left\{ \xi \mid \sum_{j=2}^n \xi_j^2 \leq \frac{C}{m} \right\}, \quad (3.5)$$

where  $m = 8n$ . We calculate  $\det(v_{ij})$ . For any point  $\xi$ , By taking an orthogonal transformation of  $\xi_2, \dots, \xi_n$ , we may assume that  $\xi = (\xi_1, \xi_2, 0, \dots, 0)$ . By a direct calculation we have

$$\begin{aligned} v_{11} &= -v \left[ -\left( \frac{\alpha}{\xi_1} - \frac{\beta}{C - \xi_1} \right)^2 + \frac{\alpha}{\xi_1^2} + \frac{\beta}{(C - \xi_1)^2} \right], \\ v_{12} &= -v \left( \frac{\alpha}{\xi_1} - \frac{\beta}{C - \xi_1} \right) \frac{2\beta\xi_2}{C - \xi_2^2}, \quad v_{ij} = 0, \quad i > 2, \quad i \neq j. \\ v_{22} &= -v \left[ \frac{2\beta(C + \xi_2^2)}{(C - \xi_2^2)^2} - \frac{4\beta^2\xi_2^2}{(C - \xi_2^2)^2} \right], \quad v_{ii} = -v \frac{2\beta}{C - \xi_2^2}, \quad i > 2. \end{aligned}$$

Denote  $A - B = v_{11}v_{22} - v_{12}^2$ ,  $D = \prod_{i=3}^n v_{ii}$ . The determinant of  $(v_{ij})$  is  $\det(v_{ij}) = (A - B) \cdot D$ . A direct calculation gives us

$$\begin{aligned} A - B &= \frac{2\beta v^2}{\xi_1^2(C - \xi_1)^2(C - \xi_2^2)^2} [\alpha(C - \xi_1)^2((1 - \alpha)C + (1 - 2\beta - \alpha)\xi_2^2) \\ &\quad + \beta\xi_1^2((1 - \beta)C + (1 - 3\beta)\xi_2^2) + 2\alpha\beta\xi_1(C - \xi_1)(C + \xi_2^2)] \\ D &= \prod_{i=3}^n v_{ii} = \left[ -v \frac{2\beta}{C - \xi_2^2} \right]^{n-2}. \end{aligned}$$

For any  $\alpha, \beta$  satisfy  $\frac{1}{2n} \leq \alpha, \beta \leq 1 - \frac{1}{2n}$ , by  $m > 4(2n - 1)$ , we have

$$A - B \geq \frac{\alpha\beta v^2}{\xi_1^2(C - \xi_2^2)^2} \frac{C(2n - 1)}{2nm}. \quad (3.6)$$

It is easy to check that  $v$  is strictly convex and

$$\det(v_{ij}) > C(n)\xi_1^{n\alpha-2}. \quad (3.7)$$

□

Now we prove

**Lemma 3.3.** *Let  $\Delta$  be a bounded open polytope in  $\mathbb{R}^n$  and  $\mathbb{D} > 0$ ,  $A$  be two smooth functions on  $\bar{\Delta}$ . Let  $u \in \mathcal{C}$  be a strictly convex function satisfying the generalized Abreu Equation (1.1). Suppose that  $\mathbb{F} = 0$  on  $\partial\Delta$ . Let  $E$  be an edge of  $\Delta$ . Suppose that  $E$  is given by  $\xi_1 = 0$ . Let  $p \in E^\circ$ . Then the following estimate holds in a neighborhood of  $p$*

$$\det(u_{ij}) \geq \frac{b}{\xi_1}$$

for some constant  $b > 0$  depending only on  $n$ ,  $\text{diam}(\Delta)$ ,  $\max_{\bar{\Delta}} \mathbb{D}$ ,  $\min_{\bar{\Delta}} \mathbb{D}$  and  $\|A\|_{L^\infty(\Delta)}$ .

*Proof.* First we prove that there exists a constant  $b_0 > 0$  such that

$$\det(u_{ij}) \geq b_0 \xi_1^{-(1-\frac{1}{n})}. \quad (3.8)$$

Choose  $\beta = \frac{1}{2}$  in Lemma 3.2. Let  $C > 0$  and  $m = 8n$  be constants such that (3.5) holds. We discuss two cases.



**Case 1.**  $n = 2$ . We choose  $\alpha = \frac{1}{2}$  and consider the following function

$$h = \mathbb{F} + b_1 v.$$

Obviously,  $h < 0$  on  $\partial\Delta$ . We have

$$\begin{aligned} \sum U^{ij} h_{ij} &= -A\mathbb{D} + b_1 \det(u_{ij}) \sum u^{ij} v_{ij} \\ &\geq -A\mathbb{D} + nb_1 \det(u_{ij})^{1-1/2} (\det(v_{ij}))^{1/2} \\ &\geq -A\mathbb{D} + nb_1 d_1 C(\epsilon_0). \end{aligned}$$

Here we use the estimate  $\det(D^2 u) \geq d_1$ . By choosing the constant  $b_1$  large, we have  $\sum U^{ij} h_{ij} \geq 0$ . So  $h$  attains its maximum on  $\partial\Delta$ . Then  $w \leq b_1 \mathbb{D}^{-1} |v|$ . It follows that

$$\det(u_{ij}) \geq b_2 \xi_1^{-\frac{1}{2}}.$$

for some constant  $b_2$ .

**Case 2.**  $n \geq 3$ . Choose a sequence  $\{\alpha_k\}$  such that

$$\alpha_k = 2 \left(1 - \left(1 - \frac{1}{n}\right)^k\right), \quad \forall k \geq 1.$$

Obviously,

$$\alpha_k - \frac{2}{n} = \left(1 - \frac{1}{n}\right) \alpha_{k-1}, \quad k \geq 2, \quad (3.9)$$

and there is  $k^* \in \mathbb{Z}^+$  such that  $\alpha_{k^*} < 1 - \frac{1}{n}$  and  $\alpha_{k^*+1} \geq 1 - \frac{1}{n}$ .

We first let  $\alpha = \alpha_1$ ,  $h = \mathbb{F} + b_2 v$ . By the same argument as in Case 1 we get

$$\det(u_{ij}) \geq b'_2 \xi_1^{-\frac{2}{n}}.$$

Next we let  $\alpha = \alpha_2$ ,  $h = \mathbb{F} + b_3 v$ . Then

$$\begin{aligned} \sum U^{ij} h_{ij} &\geq -A\mathbb{D} + nb_3 \det(u_{ij})^{1-1/n} (\det(v_{ij}))^{1/n} \\ &\geq -A\mathbb{D} + nb_3 b_{2'}^{1-\frac{1}{n}} \xi_1^{-\alpha_1(1-\frac{1}{n})+\alpha_2-\frac{2}{n}} \geq -A\mathbb{D} + nb_3 b_{2'}^{1-\frac{1}{n}}. \end{aligned}$$

We choose  $b_3$  such that  $\sum U^{ij} h_{ij} > 0$ . Then we have

$$\det(u_{ij}) \geq b'_3 \xi_1^{-\alpha_2}.$$

We iterate the process to improve the estimate. After finite many steps we get  $\det(u_{ij}) \geq b' \xi_1^{-\alpha_{k^*}}$ . Then we set  $\alpha = 1 - \frac{1}{n}$ , and repeat the argument above to get (3.8).

Next we consider the function

$$v' = \xi_1^\alpha \left( C + \sum_{j=2}^n \xi_j^2 \right) - a\xi_1,$$

where  $a > 0$ ,  $\alpha > 1$  are constants,  $C > 0$  is the constant as before. We choose  $a$  large such that  $v' \leq 0$  on  $\Delta$ . For any point  $\xi$  we may assume that  $\xi = (\xi_1, \xi_2, 0, \dots, 0)$ . By a direct calculation we have

$$\begin{aligned} v'_{11} &= \alpha(\alpha - 1)\xi_1^{\alpha-2}(C + \xi_2^2), \\ v'_{ii} &= 2\xi_1^\alpha \quad i \geq 2, \quad v'_{12} = 2\alpha\xi_2\xi_1^{\alpha-1}, \\ \det(D^2v') &= 2^{n-1} [\alpha(\alpha - 1)(C + \xi_2^2) - 2\alpha^2\xi_2^2] \xi_1^{n\alpha-2}. \end{aligned}$$

Then for large  $C$ , we conclude that  $v'$  is convex and

$$\det(D^2v') \geq C_1 \xi_1^{n\alpha-2}. \quad (3.10)$$

Set  $\alpha = 1 + \frac{1}{n^2}$ . Consider the function

$$h' = \mathbb{F} + b_5 v'.$$

Obviously,  $h' < 0$  on  $\partial\Delta$ . We have

$$\begin{aligned} \sum U^{ij} h_{ij} &= -A\mathbb{D} + b_5 \det(u_{ij}) \sum u^{ij} v'_{ij} \\ &\geq -A\mathbb{D} + nb_5 \det(u_{ij})^{1-1/n} \det(v'_{ij})^{1/n} \\ &\geq -A\mathbb{D} + nb_5 C(n) \xi_1^{-(1-\frac{1}{n})^2} C_1 \xi_1^{\alpha-\frac{2}{n}} \\ &= -A\mathbb{D} + nb_5 C(n) C_1. \end{aligned}$$

We choose  $b_5$  such that  $\sum U^{ij} h_{ij} \geq 0$ . By the maximum principle we have  $w \leq C_5 \mathbb{D}^{-1} |v'| \leq aC_5 \xi_1$ . It follows that  $\det(u_{ij})(\xi) \geq C_5 \xi_1^{-1}$  for some constant  $C_5 > 0$  independent of  $p$ .  $\square$

### 3.2. The upper bound of the determinant

Let  $u \in \mathbf{S}_{p_o}$  be a solution of the generalized Abreu Equation (1.1). In this section, we derive a global upper bound of the determinant of the Hessian of  $u$ . The proof of the following lemma is standard

**Lemma 3.4.** *Suppose that  $u \in \mathbf{S}_{p_o}$  satisfies the generalized Abreu Equation (1.1). Assume that the section*

$$\bar{S}_u(p_o, C) = \{\xi \in \Delta : u(\xi) \leq C\}$$

*is compact and that there is a constant  $b > 0$  such that*

$$\sum_{k=1}^n \left( \frac{\partial u}{\partial \xi_k} \right)^2 \leq b \quad \text{on } \bar{S}_u(p_o, C).$$

*Then,*

$$\det(u_{ij}) \leq C_2 \quad \text{in } S_u(p_o, C/2),$$

*where  $C_2$  is a positive constant depending on  $n$ ,  $C$  and  $b$ .*

Following [8] we derive a global estimate for the upper bound of  $\det(u_{ij})$  for the generalized Abreu Equation (1.1). This upper bound relates to the Legendre transforms of solutions.

For any point  $p$  on  $\partial\Delta$ , there is an affine coordinate  $\{\xi_1, \dots, \xi_n\}$ , such that, for some  $1 \leq m \leq n$ , a neighborhood  $U \subset \bar{\Delta}$  of  $p$  is defined by  $m$  inequalities

$$\xi_1 \geq 0, \quad \dots, \quad \xi_m \geq 0,$$

with  $\xi(p) = 0$ . Then,  $v$  in (2.1) has the form

$$v = \sum_{i=1}^m \xi_i \log \xi_i + \alpha(\xi),$$

where  $\alpha$  is a smooth function in  $\bar{U}$ . By Proposition 2 in [11], we have the following result.

**Lemma 3.5.** *There holds*

$$\det(v_{ij}) = [\xi_1 \xi_2 \dots \xi_m \beta(\xi)]^{-1} \quad \text{in } \Delta,$$

*where  $\beta(\xi)$  is smooth up to the boundary and  $\beta(0) = 1$ .*

For any  $q \in \Delta$  denote by  $d_E(q, \partial\Delta)$  the Euclidean distance from  $q$  to  $\partial\Delta$ . By Lemma 3.5, we have

$$\det(v_{ij}) \leq \frac{C}{[d_E(p, \partial\Delta)]^n} \quad \text{in } \Delta, \tag{3.11}$$

where  $C$  is a positive constant.

Recall that  $p_o \in \Delta$  is the point we fixed for  $\mathcal{S}_{p_o}$ . Now we choose coordinates  $\xi_1, \dots, \xi_n$  such that  $\xi(p_o) = 0$ . Set

$$x_i = \frac{\partial u}{\partial \xi_i}, \quad f = \sum_i x_i \xi_i - u.$$

**Lemma 3.6.** *Let  $\Delta$  be a bounded open polytope in  $\mathbb{R}^n$  and  $\mathbb{D} > 0$ ,  $A$  be smooth functions on  $\bar{\Delta}$ . Let  $u \in \mathbf{S}_{p_o}$  be a strictly convex function satisfying the generalized Abreu Equation (1.1). Assume, for some positive constants  $d$  and  $b$ ,*

$$\frac{1 + \sum x_i^2}{(d + f)^2} \leq b \quad \text{in } \mathbb{R}^n.$$

Then,

$$\exp \{-C_3 f\} \frac{\det(u_{ij})}{(d + f)^{2n}} \leq C_4 \quad \text{in } \Delta,$$

where  $C_3$  is a positive constant depending only on  $n$  and  $\Delta$ , and  $C_4$  is a positive constant depending only on  $n, d, b, \mathbb{D}$  and  $\max_{\bar{\Delta}} |A|$ .

*Proof.* Let  $v$  be given as in (2.1). By adding a linear function, we assume that  $v$  is also normalized at  $p_o$ . Denote  $g = L(v)$ . By (3.11), it is straightforward to check that there exists a positive constant  $C_1$  such that

$$\det(v_{ij}) e^{-C_1 g} \rightarrow 0 \quad \text{as } p \rightarrow \partial \Delta.$$

Since  $u = v + \phi$  for some  $\phi \in C^\infty(\bar{\Delta})$ , then

$$\det(u_{ij}) e^{-C_1 f} \rightarrow 0 \quad \text{as } p \rightarrow \partial \Delta. \tag{3.12}$$

Consider the function for some constant  $\varepsilon$  to be determined,

$$\mathcal{F} = \exp \left\{ -C_1 f + \varepsilon \frac{1 + \sum x_i^2}{(d + f)^2} \right\} \frac{1}{\mathbb{F} (d + f)^{2n}},$$

where  $\mathbb{F}$  is defined in (3.1),  $\varepsilon$  is a positive number to be determined latter. Obviously,  $\mathcal{F} \rightarrow 0$  as  $p \in \partial \Delta$ . Assume  $\mathcal{F}$  attains its maximum at an interior point  $p^*$ . Then at  $p^*$ , we have

$$\frac{\partial}{\partial x_j} \mathcal{F} = 0, \quad \sum f^{ij} \frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j} \leq 0.$$

Thus,

$$-(\log \mathbb{F})_i - C_1 f_i - \frac{2n f_i}{d+f} + \varepsilon \frac{1 + \sum x_k^2}{(d+f)^2} \left[ \frac{(\sum x_k^2)_i}{1 + \sum x_k^2} - 2 \frac{f_i}{d+f} \right] = 0, \quad (3.13)$$

and

$$\begin{aligned} & \sum_{i,j} f^{ij} (\log \mathbb{D})_i (\log \mathbb{F})_j + A - C_1 n - \frac{2n^2}{d+f} + \frac{2n \|\nabla f\|^2}{(d+f)^2} \\ & + \varepsilon \frac{1 + \sum x_k^2}{(d+f)^2} \left[ \frac{2 \sum_k f^{kk}}{1 + \sum x_k^2} - \frac{\|\nabla \sum x_k^2\|^2}{(1 + \sum x_k^2)^2} - \frac{2n}{d+f} + \frac{2 \|\nabla f\|^2}{(d+f)^2} \right] \\ & + \varepsilon \frac{1 + \sum x_k^2}{(d+f)^2} \left\| \left( \frac{\nabla (\sum x_k^2)}{1 + \sum x_k^2} - \frac{2 \nabla f}{d+f} \right) \right\|^2 \leq 0, \end{aligned} \quad (3.14)$$

where we used (3.3) and denote  $F_i = \frac{\partial F}{\partial x^i}$ ,  $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$  for any function  $F$ . Since

$$\sum \left| \frac{\partial \log \mathbb{D}}{\partial \xi_i} \right| \leq C,$$

and

$$\sum_{i,j} f^{ij} \frac{\partial \log \mathbb{D}}{\partial x_i} \frac{\partial \log \mathbb{F}}{\partial x_j} = \sum_{i,j,k} f^{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \log \mathbb{D}}{\partial \xi_k} \frac{\partial \log \mathbb{F}}{\partial x_i} = \sum_i \frac{\partial \log \mathbb{D}}{\partial \xi_i} \frac{\partial \log \mathbb{F}}{\partial x_i},$$

we have

$$\left| \sum_{i,j} f^{ij} (\log \mathbb{D})_i (\log \mathbb{F})_j \right| \leq C \sum |(\log \mathbb{F})_j|.$$

By  $\left| \frac{\partial f}{\partial x_i} \right| = |\xi_i| \leq \text{diam}(\Delta)$ ,  $\frac{\sum x_k^2}{(d+f)^2} \leq b$  and (3.13) we have, at  $p^*$ ,

$$\left| \sum_{i,j} f^{ij} (\log \mathbb{D})_i (\log \mathbb{F})_j \right| \leq C \sum |(\log \mathbb{F})_j| \leq C_3. \quad (3.15)$$

where  $C_3$  is the constant depending only on  $b, \text{diam}(\Delta)$  and  $n$ . Inserting (3.15) into (3.14), we obtain

$$\begin{aligned} & \varepsilon \frac{1 + \sum x_k^2}{(d+f)^2} \left[ \frac{2 \sum f^{ii}}{1 + \sum x_k^2} - \frac{4 \langle \nabla \sum x_k^2, \nabla f \rangle}{(1 + \sum x_k^2)(d+f)} - \frac{2n}{d+f} + \frac{6 \|\nabla f\|^2}{(d+f)^2} \right] \\ & + \frac{2n \|\nabla f\|^2}{(d+f)^2} - \frac{2n^2}{d+f} + A - C_2 n - C_3 \leq 0. \end{aligned} \quad (3.16)$$

By the Schwarz inequality, we have

$$\left| \frac{4\langle \nabla \sum x_k^2, \nabla f \rangle}{(1 + \sum x_k^2)(d + f)} \right| \leq \frac{\|\nabla \sum x_k^2\|^2}{4(1 + \sum x_k^2)^2} + \frac{16\|\nabla f\|^2}{(d + f)^2}.$$

Hence,

$$\left| \frac{4\langle \nabla \sum x_k^2, \nabla f \rangle}{(1 + \sum x_k^2)(d + f)} \right| \leq \frac{\sum f^{ii}}{(1 + \sum x_k^2)^2} + \frac{16\|\nabla f\|^2}{(d + f)^2}. \quad (3.17)$$

Combining (3.16) and (3.17) yields

$$\begin{aligned} & \varepsilon \frac{1 + \sum x_k^2}{(d + f)^2} \left[ \frac{\sum f^{ii}}{1 + \sum x_k^2} - \frac{2n}{d + f} - \frac{10\|\nabla f\|^2}{(d + f)^2} \right] \\ & + \frac{2n\|\nabla f\|^2}{(d + f)^2} - \frac{2n^2}{d + f} + A - C_2n - C_3 \leq 0. \end{aligned}$$

By choosing  $\varepsilon > 0$  such that  $10\varepsilon b \leq 1$ , we have

$$\varepsilon \frac{\sum f^{ii}}{(d + f)^2} + A - C_4 \leq 0.$$

By the relation between the geometric mean and the arithmetic mean, we get

$$\frac{\det(u_{ij})}{(d + f)^{2n}} = \frac{(\det(f^{ij}))^{-1}}{(d + f)^{2n}} \leq \frac{C(n)(\sum f^{ii})^n}{(d + f)^{2n}} \leq C_5.$$

Therefore,  $\mathcal{F}(p^*) \leq C_6$ , and hence  $\mathcal{F} \leq C_6$  everywhere. The definition of  $\mathcal{F}$  and the bound of  $\mathbb{D}$  implies

$$\exp\{-C_2f\} \frac{\det(u_{ij})}{(d + f)^{2n}} \leq C_7.$$

This is the desired estimate.  $\square$

#### 4. Convergence theorems in section

Let  $\Omega^* \subset \mathbb{R}^n$ . Denote by  $\mathcal{F}(\Omega^*, C)$  the class of smooth convex functions defined on  $\Omega^*$  such that

$$\inf_{\Omega^*} u = 0, \quad u = C > 0 \quad \text{on} \quad \partial\Omega^*.$$

**Lemma 4.1.** *Let  $\Omega^* \subset \mathbb{R}^n$  be a normalized domain,  $u \in \mathcal{F}(\Omega^*, C)$  be a function satisfying the generalized Abreu Equation (1.1). Suppose that there is a constant  $C_1 > 0$  such that in  $\Omega^*$*

$$C_1^{-1} \leq \det(u_{ij}) \leq C_1. \quad (4.1)$$

*Then for any  $\Omega^\circ \subset\subset \Omega^*$ ,  $p > 2$ , we have the estimate*

$$\|u\|_{W^{4,p}(\Omega^\circ)} \leq C, \quad \|u\|_{C^{3,\alpha}(\Omega^\circ)} \leq C, \quad (4.2)$$

*where  $C$  depends on  $n, p, \mathbb{D}, C_1, \|A\|_{L^\infty(\Delta)}, \text{dist}(\Omega^\circ, \partial\Omega^*)$ .*

**Proof.** In [3] Caffarelli-Gutierrez proved a Hölder estimate of  $\det(u_{ij})$  for homogeneous linearized Monge-Ampère equations assuming that the Monge-Ampère measure  $\mu[u]$  satisfies some condition, which is guaranteed by (4.1). Consider the generalized Abreu Equation

$$\sum U^{ij} \mathbb{F}_{ij} = -A\mathbb{D}, \quad \mathbb{F} := \frac{\mathbb{D}}{\det(u_{ij})}$$

where  $A \in L^\infty(\Omega)$ . Since  $\mathbb{D} \in C^\infty(\bar{\Delta})$  and  $\mathbb{D} > 0$ , by the same argument in [3] one can obtain the Hölder continuity of  $\det(u_{ij})$ . Then Caffarelli's  $C^{2,\alpha}$  estimates for Monge-Ampère equations [2] give us

$$\|u\|_{C^{2,\alpha}(\Omega^*)} \leq C_2.$$

Following from the standard elliptic regularity theory we have  $\|u\|_{W^{4,p}(\Omega^*)} \leq C_3$ . By the Sobolev embedding theorem

$$\|u\|_{C^{3,\alpha}(\Omega^\circ)} \leq C_4 \|u\|_{W^{4,p}(\Omega^*)}.$$

Then the lemma follows. ■

Let  $\Omega \subset \mathbb{R}^n$ . Denote by  $\mathcal{F}(\Omega, C)$  the class of smooth convex functions defined on  $\Omega$  such that

$$\inf_{\Omega} f = 0, \quad f = C > 0 \text{ on } \partial\Omega.$$

Next we prove the following convergence theorem.

**Theorem 4.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a normalized domain. Let  $f_{(k)} \in \mathcal{F}(\Omega, C)$  be a sequence of functions satisfying the equation*

$$-\sum_{i,j} f_{(k)}^{ij} \frac{\partial^2(\log \mathbb{F}_{(k)})}{\partial x_i \partial x_j} - \sum_{i,j} f_{(k)}^{ij} \frac{\partial(\log \mathbb{F}_{(k)})}{\partial x_i} \frac{\partial(\log \mathbb{D})}{\partial x_j} = A_{(k)}. \quad (4.3)$$

*Suppose that  $A_{(k)}$   $C^m$ -converges to  $A$  on  $\bar{\Omega}$  and there are constants  $0 < C_1 < C_2$  independent of  $k$  such that*

$$C_1 \leq \det \left( \frac{\partial^2 f_{(k)}}{\partial x_i \partial x_j} \right) \leq C_2 \quad (4.4)$$

*hold in  $\Omega$ . Then there exists a subsequence of functions, without loss of generality, still denoted by  $f_{(k)}$ , locally uniformly converging to a function  $f_\infty$  in  $\Omega$  and, for any open set  $\Omega_o$  with  $\bar{\Omega}_o \subset \Omega$ , and for any  $\alpha \in (0, 1)$ ,  $f_{(k)}$   $C^{m+3,\alpha}$ -converges to  $f_\infty$  in  $\Omega_o$ .*

**Proof.** It is obvious that there exists a subsequence of functions, locally uniformly converging to a function  $f_\infty$  in  $\Omega$ . A fundamental result on Monge-Ampère equation tell us that  $f_\infty$  is  $C^{1,\alpha}$  and strictly convex (see [17]). Suppose that  $f_\infty(p) = 0$  for some point  $p \in \Omega$ . We choose the coordinates  $x = (x_1, \dots, x_n)$  such that  $x(p) = 0$ . Put

$$u_{(k)} = \sum x_i \frac{\partial f_{(k)}}{\partial x_i} - f_{(k)}, \quad \Omega_{(k)}^* = \nabla f_{(k)}(\Omega),$$

$$u_\infty = \sum x_i \frac{\partial f_\infty}{\partial x_i} - f_\infty, \quad \Omega_\infty^* = \nabla f_\infty(\Omega).$$

We have  $f_\infty(0) = u_\infty(0) = 0$ . The key point of the proof of the Theorem is the following

**Claim.** There are constants  $C > 0$ ,  $b > r > 0$  such that  $\bar{S}_{u_\infty}(0, C)$  is compact and

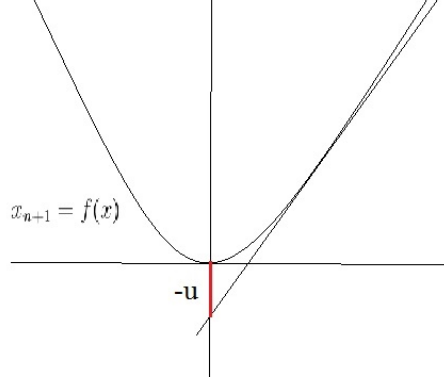
$$D_r(0) \subset S_{u_\infty}(0, C) \subset D_b(0).$$

$$x_{n+1} = f(x)$$

**Proof of Claim.** Denote

$$M := \{(x, f_\infty(x)) | x \in \Omega\}, \quad M^* := \{(\xi, u_\infty(\xi)) | \xi \in \Omega_\infty^*\}.$$





Then  $M$  is a  $C^{1,\alpha}$  strictly convex hypersurface with the support hyperplane  $H = \{x | x_{n+1} = 0\}$  at 0. We look at the geometry meaning of  $u_\infty$ . Let  $q \in \Omega$  be a point near 0. The support hyperplane of  $M$  at  $(q, f_\infty(q))$  is given by

$$H_{(q, f_\infty(q))} = \left\{ (x_1, \dots, x_n, x_{n+1}) \mid \sum x_i \frac{\partial f}{\partial x_i}(q) + x_{n+1} = \sum x_i(q) \frac{\partial f}{\partial x_i}(q) + f(q) \right\}.$$

The intersection

$$H_{(q, f_\infty(q))} \cap \{(0, \dots, 0, x_{n+1})\} = (0, \dots, 0, -u_\infty(q)). \quad (4.5)$$

In particular, we have

( $\star$ )  $u_\infty$  is monotonically increase along every ray from 0:  $\{x_i = a_i t, t \geq 0, i = 1, \dots, n\}$ , where  $a_i$  are constants with  $\sum a_i^2 = 1$ .

By strictly convexity of  $f_\infty$  we can find  $b_1 > b_2 > 0, d_1 > d_2 > 0$  such that

$$(1) \quad \bar{S}_{f_\infty}(0, b_2) \subset \bar{S}_{f_\infty}(0, b_1) \subset D_{d_1}(0) \subset \Omega,$$

$$(2) \quad \text{dist}(\bar{S}_{f_\infty}(0, b_2), \partial \bar{S}_{f_\infty}(0, b_1)) \geq d_2.$$

Then

$$|\nabla f_\infty| \leq \frac{b_1}{d_1} \quad \forall x \in \bar{S}_{f_\infty}(0, b_2).$$

It follows that  $\bar{S}_{u_\infty}(0, C) \subset D_b(0)$  for some constant  $b > 0$ . By compactness we can find  $p \in \partial S_{f_\infty}(0, b_2)$  such that  $u_\infty(p) = \min_{\partial S_{f_\infty}(0, b_2)} \{u_\infty\}$ . By ( $\star$ ) we can find a set  $\Omega^\circ \subset \bar{S}_{f_\infty}(0, b_2)$  such that

$$u_\infty(x) = u_\infty(p) \quad \forall x \in \partial \Omega^\circ.$$

Let  $q \in \partial\Omega^\circ$  be the point with  $f_\infty(q) = \min_{\partial\Omega^\circ} \{f_\infty\}$ . By the strictly convexity of  $f_\infty$ , we have  $f_\infty(q) > 0$ . Consider the convex cone  $V$  with vertex  $(0, 0)$  and the base

$$\{(x_1, x_2, \dots, x_n, f_\infty(q)) | x_1^2 + \dots + x_n^2 = d_1^2\}.$$

By the comparison theorem of normal maps, there exists a Euclidean ball  $D_r(0)$  such that  $D_r(0) \subset \nabla f_\infty(\bar{S}_{f_\infty}(0, f_\infty(q)))$ . We choose  $C = u_\infty(p)$ . Then

$$D_r(0) \subset S_{u_\infty}(0, C) \subset D_b(0).$$

The claim follows.

By the claim we conclude that

$$\bar{S}_{u_{(k)}}(0, C/2) := \{\xi | u_{(k)} \leq C/2\}$$

is compact and contain a Euclidean ball for  $k$  large enough. By (4.4) we have

$$\frac{1}{C_2} \leq \det \left( \frac{\partial^2 u_{(k)}}{\partial \xi_i \partial \xi_j} \right) \leq \frac{1}{C_1} \quad (4.6)$$

A direct calculation shows that  $u_{(k)}$  satisfy the generalized Abreu Equation (1.1). By Lemma 4.1  $u_{(k)}$   $C^{m+3}$ -converges to  $u_\infty$ . It follows that  $f_{(k)}$   $C^{m+3}$ -converges to  $f_\infty$  in a neighborhood of 0.

Now let  $p \in \Omega$  be an arbitrary point, let  $l$  be the linear function defining the support hyperplane of  $M$  at  $(p, f_\infty(p))$ . Let

$$\tilde{f}_\infty = f_\infty - l.$$

We use  $\tilde{f}_\infty$  instead  $f$  and use the same argument above. The theorem follows.  $\blacksquare$

## 5. Proof of the Main Theorem

Since  $(\Delta, \mathbb{D}, A)$  is uniformly  $K$ -stable and  $A^{(k)}$  converges to  $A$  smoothly in  $\bar{\Delta}$ , then  $(\Delta, \mathbb{D}, A^{(k)})$  is uniformly  $K$ -stable for large  $k$ , i.e.,  $\Delta$  is  $(A^{(k)}, \mathbb{D}, \lambda)$ -stable for some constant  $\lambda > 0$  independent of  $k$ . Since  $u^{(k)}$  satisfies the generalized Abreu Equation (1.1), then

$$\mathcal{L}_{A_k}(u^{(k)}) = \int_{\Delta} \sum_{i,j} (u^{(k)})^{ij} (u^{(k)})_{ij} \mathbb{D}d\mu = n \int_{\Delta} \mathbb{D}d\mu$$

and hence,

$$\int_{\partial\Delta} u d\sigma \leq n\lambda^{-1} \frac{\max_{\Delta} \mathbb{D}}{\min_{\Delta} \mathbb{D}} \text{Area}(\Delta).$$

It follows that  $u^{(k)}$  locally and uniformly converges to a convex function  $u$  in  $\Delta$ .

*Claim.* For any point  $\xi \in \Delta$  and any  $B_\delta(\xi) \subset \Delta$ , there exists a point  $\xi_o \in B_\delta(\xi)$  such that  $u$  has second derivatives and is strictly convex at  $\xi_o$ . Here,  $B_\delta(\xi)$  denotes the Euclidean ball centered at  $\xi$  with radius  $\delta$ .

The proof of the claim is the same as in [4], see also [8].

We now choose coordinates such that  $\xi_o = 0$ . By adding linear functions, we assume that all  $u^{(k)}$  and  $u$  are normalized at 0. Since  $u$  is strictly convex at 0, there exist constants  $\epsilon' > 0$ ,  $d_2 > d_1 > 0$  and  $b' > 0$ , independent of  $k$ , such that, for large  $k$ ,

$$B_{d_1}(0) \subset \bar{S}_{u^{(k)}}(0, \epsilon') \subset B_{d_2}(0) \subset \Delta,$$

and

$$\sum_i \left( \frac{\partial u^{(k)}}{\partial \xi_i} \right)^2 \leq b' \quad \text{in } S_{u^{(k)}}(0, \epsilon').$$

By Lemma 3.1 and Lemma 3.4, we have

$$C_1 \leq \det(u_{ij}^{(k)}) \leq C_2 \quad \text{in } S_{u^{(k)}}(0, \frac{1}{2}\epsilon'), \quad (5.1)$$

where  $C_1 < C_2$  are positive constants independent of  $k$ . By Lemma 4.1  $\{u^{(k)}\}$  converges smoothly to  $u$ . Therefore,  $u$  is a smooth and strictly convex function in  $S_u(0, \epsilon'/2)$ .

Let  $f^{(k)}$  be the Legendre transform of  $u^{(k)}$ . Then,  $\{f^{(k)}\}$  locally uniformly converges to a convex function  $f$  defined in the whole  $\mathbb{R}^n$ . Furthermore, in a neighborhood of 0,  $f$  is a smooth and strictly convex function such that its Legendre transform  $u$  satisfies the generalized Abreu Equation (1.1). By the convexity of  $f^{(k)}$  and the local and uniform convergence of  $\{f^{(k)}\}$  to  $f$ , we conclude, for any  $k$ ,

$$\frac{1 + \sum_i x_i^2}{(d + f^{(k)})^2} \leq b \quad \text{in } \mathbb{R}^n,$$

and, for any  $C > 1$ ,

$$B_r(0) \subset S_{f^{(k)}}(0, C) \subset B_{R_C}(0),$$

for some positive constants  $d, b, r$  and  $R_C = R(C) > 0$ . By Lemma 3.1 and Lemma 3.6, we have

$$\exp\{-C_3 C\} \frac{1}{(d+C)^{2n}} \leq \det(f_{ij}^{(k)}) \leq C_1.$$

We note that each  $f^{(k)}$  satisfies (4.3). By Theorem 4.2 we conclude that  $\{f^{(k)}\}$  uniformly and smoothly converges to  $f$  in  $S_f(0, C/2)$ . Since  $C$  is arbitrary,  $f$  is a smooth and strictly convex function in  $\mathbb{R}^n$ , and the sequence  $\{f^{(k)}\}$  locally and smoothly converges to  $f$ . By Legendre transforms, we obtain that  $u$  is a smooth and strictly convex function in  $\Delta$  and that the sequence  $\{u^{(k)}\}$  locally and smoothly converges to  $u$ . This completes the proof of Theorem 1.2.

## References

- [1] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math., 9(1998), 641-651.
- [2] L. A. Caffarelli, *Interior  $W^{2,p}$  estimates for solutions of Monge-Ampère equations*, Ann. Math., 131(1990), 135-150.
- [3] L. A. Caffarelli, C. E. Gutiérrez, *Properties of the solutions of the linearized Monge-Ampère equations*, Amer. J. Math., 119(1997), 423-465.
- [4] B. Chen, A.-M. Li, L. Sheng, *The Abreu equation with degenerated boundary conditions*, J. Diff. Equations, 252(2012), 5235-5259.
- [5] B. Chen, A.-M. Li, L. Sheng, *Interior regularity on the Abreu equation*, Acta Mathematica Sinica, 29(2013), 33-38.
- [6] B. Chen, A.-M. Li, L. Sheng, *Affine techniques on extremal metrics on toric surfaces*, arXiv:1008.2606.
- [7] B. Chen, A.-M. Li, L. Sheng, *Extremal metrics on toric surfaces*, arXiv:1008.2607.

- [8] B.H. Chen, Q. Han, A.-M. Li, L. Sheng, *Interior Estimates for the  $n$ -dimensional Abreu's Equation*, Advances in mathematics, 251(2014), 35-46
- [9] B.-H. Chen, A.-M. Li, L. Sheng, *Uniform  $K$ -stability for extremal metrics on toric varieties*, J. Diff. Equations, 257(2014), 1487-1500, arXiv:1109.5228v2.
- [10] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Diff. Geom., 62(2002), 289-349.
- [11] S. K. Donaldson, *Interior estimates for solutions of Abreu's equation*, Collect. Math., 56(2005), 103-142.
- [12] S. K. Donaldson, *Extremal metrics on toric surfaces: a continuity method*, J. Diff. Geom., 79(2008), 389-432
- [13] S. K. Donaldson, *Constant scalar curvature metrics on toric surfaces*, Geom. Funct. Anal., 19(2009), 83-136.
- [14] S. K. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, Handbook of Geometric Analysis, No. 1, International Press, Boston, 2008.
- [15] R. Feng, G. Székelyhidi, *Periodic solutions of Abreu's equation*, Matt. Res. Lett., 18(2011), 1271-1279.
- [16] V. Guillemin, *Kähler structures on toric varieties*, J. Diff. Geom., 40(1994), 285-309.
- [17] C.E. Guitiérrez, *The Monge-Ampère Equation*, Birkhause, Boston, MA, 2001.
- [18] A.-M. Li, R. Xu, U. Simon, F. Jia, *Affine Bernstein Problems and Monge-Ampère Equations*, World Scientific, 2010.
- [19] A.-M. Li, F. Jia, *A Bernstein properties of some fourth order partial differential equations*, Result. Math., 56 (2009), 109-139.
- [20] A.-M. Li, L. Sheng, G. Zhao *Differential inequalities on homogeneous toric bundles*, Preprint

- [21] T. Nyberg, Constant Scalar Curvature of Toric Fibrations. PhD thesis.
- [22] Podesta, Spiro, Kahler-Ricci solitons on homogeneous toric bundles I, II Arxiv DG/0604070/0604071
- [23] A. Raza. Scalar curvature and multiplicity-free actions. PhD thesis.
- [24] G. Székelyhidi, *Extremal metrics and K-stability*, Bull. London Math. Soc., 39(2007), 76-84.
- [25] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math., 130(1997), 1-39.
- [26] G. Tian, *Canonical Metrics in Kähler Geometry*, Lectures in Mathematics ETH Zurich, Birkhäuser Verlag, Basel, 2000.
- [27] N. S. Trudinger, X. Wang, *Berstein-Jörgens theorem for a fourth order partial differential equation*, J. Partial Diff. Equations, 15(2002), 78-88.
- [28] S.-T. Yau, *Open problems in geometry*, Proc. Symposia Pure Math., 54(1993), 1-28.